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ASYMPTOTIC SOLUTION OF TWO-POINT BOUNDARY VALUE PROBLEMS*

L. D. AKULENKO

An estimate is obtained of solutions of the input and the averaged boundary value problems for standard systems /1—6/. Such problems occur in investigations of optimal oscillating processes, using the necessary conditions of optimality of the maximum principle /7/ (see, e.g., /8—12/). Specific formulations of problems of optimal control with mechanical content are investigated.

1. Basic assumptions and statement of the problem. The use of quantities small relative to control effects in many applied problems of optimal control of quasilinear fluctuating processes leads to the investigation of boundary value problems of the maximum principle for standard systems over a specified asymptotically large time interval. By combining the osculating phase variable with its corresponding conjugate in a single slow vector and eliminating on appropriate assumptions (see Sect.3) vectors of Lagrange coefficients /10,11/ which are linear in the transversality relations at the left- and right-hand ends, we obtain the two point boundary value problem

$$x^{\cdot} = \varepsilon X(t, x), \quad M(x(0), x(T)) = 0 \tag{1.1}$$

where x is the unkown slow n-vector; t is the independent variable (time), $t \in [0, T]$, $T = \Theta\varepsilon^{-1}$, $\Theta = \text{const} > 0$; ε is a small numerical parameter, $\varepsilon \in [0, \varepsilon_0]$; $X(t, x)$ and $M(y, z)$, are specified vector functions of dimension $n \geqslant 2$. Note that in boundary value problems of the maximum principle the boundary condition (1.1) usually consists of two relations, one at the beginning $(t = 0)$ and the other at the end $(t = T)$ of the control process

$$M_0(x(0)) = 0, \quad M_T(x(T)) = 0, \quad m_0 + m_T = n$$

which are of dimensions m_0 and m_T, respectively.

If it is formally assumed that one of the quantities m_0 or m_T is zero, Eqs. (1.1) are of the form of the Cauchy problem for whose analysis the method of averaging is widely used /1—6/. It is therefore important to establish as wide as possible conditions to be imposed on functions $X(t, x)$ and $M(y, z)$ that are sufficient for the solutions of the input (1.1) and the averaged boundary value problem to be sufficiently close.

Let us assume that functions X and M satisfy the following conditions.

1. Function $X(t, x)$ is determinate for all $t \geqslant 0$, measurable in t for fixed $x \in D_x$, where D_x is an open connected set.

2. The known requirement for the existence of an X uniform with respect to $x \in D_x$ averaged over t /1—3/ is replaced by a stricter one, namely, we assume that $X(t, x)$ is uniformly quasiperiodic in t, i.e. that it represents a finite sum of functions $X^{(i)}(t, x)$ $(i = 1, \ldots, k \geqslant 1)$ periodic in t of arbitrary constant periods Π_i.

3. Functions $X(t, x)$ and $M(y, z)$ are determinate for all $x, y, z \in D_x$ and uniformly continuous with respect to x, y, and z.

4. There exist constants C_X and $C_M > 0$ dependent on D_x such that

$$|X(t, x)| \leqslant C_X, \quad |M(y, z)| \leqslant C_M, \quad t \geqslant 0, \quad x, y, z \in D_x \tag{1.2}$$

5. Functions X and M moreover satisfy the Lipschitz conditions with respect to $x, y, z \in D_x$, i.e. that there exist constants λ_X and λ_M, generally dependent on D_x, such that

$$\begin{aligned} &|X(t, x') - X(t, x'')| \leqslant \lambda_X |x' - x''|, \quad t \geqslant 0, \quad x', \; x'' \in D_x \\ &|M(y', z') - M(y'', z'')| \leqslant \lambda_M (|y' - y''| + |z' - z''|), \quad y', y'', z', z'' \in D_x \end{aligned} \tag{1.3}$$

For simplifying the proofs the stipulations (1.3) and further strengthened by the assumption of the existence of continuous and bounded partial derivatives

$$|\partial X / \partial x| \leqslant \lambda_X, \quad |\partial M / \partial y| \leqslant \lambda_M, \quad |\partial M / \partial z| \leqslant \lambda_M \tag{1.4}$$

6. When the indicated requirements are satisfied, the mean of function $X(t, x)$ averaged over t is of the form $X_0(x$, and uniform for all $x \in D_x$

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$$X_0(x)=\lim_{S\to\infty}\frac{1}{S}\int_t^{t+S} X(s,x)\,ds=\sum_{i=1}^{k} X_0^{(i)}(x),\quad t\geqslant 0,\quad X_0^{(i)}(x)=\frac{1}{\Pi_i}\int_t^{t+\Pi_i} X^{(i)}(s,x)\,ds,\quad i=1,\ldots,k \tag{1.5}$$

Function $X_0(x)$ has the properties 3—5. It was shown in /1—6/ that solutions of the Cauchy problems in certain domains of common initial values of input and averaged equations possess the property of being ε-close to each other throughout the time interval $t\in[0,\ T]$.

Let us consider besides the input boundary value problem (1.1) the simpler averaged in conformity with (1.5) problem

$$d\xi\,/\,d\tau=X_0(\xi),\ \ M(\xi(0),\xi(\Theta))=0,\ \ \tau=\varepsilon t\in[0,\ \Theta] \tag{1.6}$$

We assume the existence of some solution $\xi(\tau)$ of the boundary value problem (1.6), which belongs to region D_x, for all $\tau\in[0,\Theta]$ and is unique.

We shall investigate the question what are the additional conditions to be imposed on functions $X(t,x)$ and $M(y,z)$ for the existence also of solution $x(t,\varepsilon)$ of the input boundary value problem, and how close that solution is to solution $\xi(\varepsilon t)$ for $t\in[0,\ \Theta\varepsilon^{-1}]$, where $\varepsilon\in(0,\ \varepsilon_0]$.

Note that the construction of function $\xi(\tau)$ is usually simpler, since the order of the integrated system of equations (1.6) is lesser by one that of (1.1), owing to the absence of dependence on t. Moreover, the system of Eqs. (1.1) derived from the maximum principle is Hamiltonian /7,10,11/. As shown in /10,11/, system (1.6) is also of Hamiltonian form with the known "energy" integral. Hence it is possible to reduce further the order of the system of differential equation by one. The numerical solution of the boundary value problem (1.6) is more convenient since integration is carried out over a shorter interval of slow time $\tau\in[0,\ \Theta],\ \Theta\sim 1$, and the equations do not contain rapid oscillations.

2. Evaluation of the closeness of solutions of the input and the averaged boundary value problems. The following constructive approach to the investigation of existence of solution $x(t,\ \varepsilon)$ of the input two-point problem (1.1) to the solution $\xi(\tau)$, $\tau=\varepsilon t$ of the averaged problem (1.6). By virtue of assumptions 1—5 in Sect.1 the Cauchy problem

$$x^{\cdot}=\varepsilon X(t,\ x),\ x(0)=a\in D_a\subset D_x,\ t\in[0,\ T] \tag{2.1}$$

has a solution of the form /3,6/

$$x(t,\ a,\ \varepsilon)=\varphi(\tau,\ a)+v(t,\ a,\ \varepsilon) \tag{2.2}$$

where D_a is a nonempty open connected set, $\varphi(\tau,\ a)$ is a function continuously differentiable with respect to $\tau\in[0,\ \Theta]$ and $a\in D_a$ which represents the general solution of the Cauchy problem (2.1)

$$d\xi\,/\,d\tau=X_0(\xi),\ \xi(0)=a\in D_a,\ \xi=\varphi(\tau,\ a) \tag{2.3}$$

According to /3,6/ function v in (2.2) is uniformly bounded with respect to t and ε, and continuous with respect to a

$$|v|\leqslant\varepsilon C_v,\ \lim_{a_k\to a_*} v(t,\ a_k,\ \varepsilon)=v(t,\ a_*,\ \varepsilon),\quad t\in[0,\ T],\ a_k,\ a_*\in D_a,\ C_v=\mathrm{const}>0,\ \varepsilon\in[0,\ \varepsilon_0] \tag{2.4}$$

The solution of the averaged boundary value problem $\xi(\tau)$, which in conformity with (1.6) is determinate and corresponds to some parameter $a=a^*\in D_a$ in function $\varphi(\tau,a)$ from (2.3). This value of a satisfies the nonlinear system

$$M(a,\ \varphi(\Theta,\ a))\equiv M^*(a)=0 \tag{2.5}$$

A similar solution $x(t,\varepsilon)$ of the input problem is obtained by substituting in (2.2) for parameter a its value determined by the relations

$$M^*(a)+N(\alpha,\ \varepsilon)=0,\ N\equiv M(a,\ \varphi(\Theta,\ a)+v(T,\ a,\ \varepsilon))-M^*(a) \tag{2.6}$$

where function N is continuous with respect to $a\in D_a$ and by virtue of (1.4) is uniformly bounded with respect to $a\in D_a:|N|\leqslant\varepsilon C_N,\ \varepsilon\in[0,\ \varepsilon_0]$.

For the investigation of the implicit function $a(\varepsilon)$ we assume that the root a^* determined by (2.5) is simple, i.e.

$$\det(\partial M^*(a^*)\,/\,\partial a)\neq 0 \tag{2.7}$$

We seek root $a\in D_a$ of Eq. (2.6) of the form $a=a^*+\alpha$, where the unknown α continuously depends on ε and vanishes when $\varepsilon=0$. It is determined by the equation

$$\alpha=-\left[\frac{\partial M^*(a^*)}{\partial a}\right]^{-1}\frac{\partial M^*(a^*,\varphi(\Theta,a^*))}{\partial\varphi}v(T,a^*,\varepsilon)+A(\alpha,\varepsilon) \tag{2.8}$$

where A is a continuous function of α, ε when $a^* + \alpha \in D_a$, $\varepsilon \in [0, \varepsilon_0]$ and identically vanishes for $\varepsilon = 0$, $\alpha = 0$, and $|A| = o(\varepsilon + |\alpha|)$.

To calculate α from (2.8) we use the method of successive approximations with A as the perturbation. This yields sequence $\alpha_{j+1} = \alpha_1 + A(\alpha_j, \varepsilon)$, $j \geqslant 1$ which for fairly small $\varepsilon > 0$ is uniformly bounded and equicontinuous. Using the Arzelà theorem /13/ it is possible to select a subsequence $\alpha_{j_l} \to \alpha^*$ uniformly converging to solution (2.8) and such that $a_{j_l} = a^* + \alpha_{j_l} \to a_* \in D_a$, $a_* = a^* + \alpha^*$, $|\alpha^*| \leqslant \varepsilon C_\alpha$.

The following theorem is therefore valid.

Theorem. When conditions (1—6) of Sect.1 and (2.5) and (2.7) are satisfied, the input boundary value problem (1.1) in the region

$$t \in [0, T], \quad x \in D_x, \ \varepsilon \in [0, \varepsilon_0] \tag{2.9}$$

admits for fairly small $\varepsilon_0 > 0$ the solution $x(t, \varepsilon)$, in the ε-neighborhood of the generating solution $\xi(\tau) = \varphi(\tau, a^*)$ of the averaged problem (1.6)

$$|x(t, \varepsilon) - \xi(\tau)| \leqslant \varepsilon C, \quad C = \text{const} > 0 \tag{2.10}$$

Moreover, system (1.1) does not admit in region (2.9) solutions that do not satisfy estimate (2.10). Constant C in estimate (2.10) can be effectively determined in terms of the problem coefficients and size of regions D_x and D_a.

Remarks. 1°. The uniqueness of solution of Eq. (2.8) and of the boundary value problem (1.1) is not guaranteed. For the existence and uniqueness of the input boundary value problem solution that becomes the generating one $\xi(\tau)$ when $\varepsilon = 0$ it is sufficient to impose in addition to condition (2.7) the requirement that function A in (2.8) must satisfy the Lipschitz condition or has a uniformly bounded derivative with respect to α. These stipulations entail the imposition on function v in (2.2) of similar conditions of smoothness with respect to the argument a. Note that an equal estimate of the quantity $\partial v / \partial a$ can be obtained for all $t \in [0, T]$, $a \in D_a$, $\varepsilon \in [0, \varepsilon_0]$ using the standard method based on the Gronwall— Bellman lemma /1,6,13/. However that requires higher smoothness properties of function X with respect to $x \in D_x$, namely, the existence of derivatives $\partial X / \partial x$ uniformly bounded and satisfying the Lipschitz condition with respect to $x \in D_x$ with constants that are independent of t and ε. These conditions of smoothness are superfluous for constructing the first approximation.

2°. The fulfillment of inequality (2.7) is essential, otherwise for $\varepsilon > 0$ branching of the root a^* of Eq. (2.6) into quantities of order of fractional power /14/ is possible. The distinction between the generating and the exact solutions is generally of the order of the lower nonzero power of the small parameter. This critical case requires additional investigation.

3°. Note that in the particular case of the terminal control problem (the quality criterion $\Phi(z(T)) \to \min$) the boundary conditions for vector $x = (z, p)$, where z and p are, respectively, the phase and the conjugate vectors of the form $z(0) = z^\circ$, $p(T) = -\Phi'(z(T))$. The inequality (2.7) is equivalent to the stipulation of the existence of a general solution $\xi = (\zeta, \eta)$ of the averaged system such that $\zeta(0, a, b) = a$, $\eta(\Theta, a, b) = b$, where parameter a belongs to the neighborhood of point z° and b to that of point $b^*(z^\circ)$ which is the simple root of the equation $\Phi'(\zeta(\Theta, z^\circ, b)) = -b$.

4°. The theorem formulated above and the preceding remarks are valid for every admissible root $a^* \in D_a$ of Eq. (2.5). Selection of the required $a(\varepsilon) = a^* + \alpha(\varepsilon)$ is based on supplementary conditions. In the problem of optimal control the root $a(\varepsilon)$ is selected from the condition of minimum of functional calculated with reasonable accuracy with respect to ε. In the formulation considered above the functional is calculated with the allowable error $O(\varepsilon)$.

Thus the above theorem validates the application of the method of averaging in problems of optimal control.

5°. It should be also pointed out that more general systems with rotating phases

$$a^\cdot = \varepsilon A(\tau, a, \psi), \ a = (a_1, \ldots, a_n), \ \tau = \varepsilon t, \ t \in [0, T], \quad \psi^\cdot = \nu(\tau) + \varepsilon\Psi(\tau, a, \psi), \ \psi = (\psi_1, \ldots, \psi_r), \ \nu_j(\tau) \geqslant \nu_0 > 0 \tag{2.11}$$

reduce to equations of the form (1.1). Thus, when $r = 1$ and functions A and Ψ are, similarly to X, quasiperiodic in ψ, the introduction of the new independent variable θ and the slow variable φ by formulas

$$\theta = \int_0^t \nu(\varepsilon s)\, ds, \quad \psi = \theta + \varphi, \quad \varepsilon\theta = \int_0^\tau \nu(\sigma)\, d\sigma \tag{2.12}$$

system (2.11) reduces to the form (1.1).

When $r > 1$ and additional requirement $\nu_1(\tau) \equiv \ldots \equiv \nu_r(\tau)$ similar to /2.12/ by vectorial substitude of system /2.11/ also lead to form (1.1). If $\nu_j = \text{const} \geqslant 0$ (ν_j are arbitrary), then at decomposability condition of functions A and Ψ into finite trigonometric sums for the system (2.11), there is possible to reduce (1.1) to standard form.

3. Application to problems of control. Let us consider the following problem of optimal control of the motion of the oscillating system:

$$q^{\cdot} = Aq + F(t) + \varepsilon\,[G(t, q)\,u + L(t, q)],\ q(0) = q^{\circ},\ S(q)|_T = 0,\quad J = \frac{\varepsilon}{2}\int_0^T u^2\,dt \to \min,\quad 0 \leqslant t \leqslant T = \Theta\varepsilon^{-1} \qquad (3.1)$$

where q is the n-vector of generalized phase coordinates, u is the m-vector of control functions, A is a constant skew-symmetric matrix, and S is a specified function of dimension $r \leqslant n$. It is assumed that F, G, and L are quasiperiodic functions of t, of the form of trigonometric polynomials with a limited set of frequencies $\{\Omega\}$, and the coefficients are polynomials in q whose power does not exceed some $k \geqslant 0$.

Let all characteristic indices—eigenvalues of matrix A—have zero real parts and the number of elementary divisors n be of the dimension of vector q. Then the elements of the fundamental matrix $Q(t)$ $(Q(0) = I)$ and of its inverse $Q^{-1}(t) = Q(-t)$ are quasiperiodic functions of t with the frequency basis $\{\nu\}$ and the determinant $|Q(t)| = 1$. Some of these frequencies may be zero. Let us assume that the unperturbed system (3.1) (when $\varepsilon = 0$) has no resonance solutions, i.e. its particular solution q^* is a bounded quasiperiodic function of t with the set of frequencies $\{\Omega - \nu, \Omega + \nu\}$

$$q^*(t) = \int_0^t Q(t-s)\,F(s)\,ds \qquad (3.2)$$

Using formula (3.2) we pass in system (3.1) to the osculating variable x

$$q = q^*(t) + Q(t)\,x \qquad (3.3)$$

Differentiating (3.3) we obtain by virtue of system (3.1) the equation of controlled motion for the slow variable x

$$x^{\cdot} = \varepsilon\,[g(t, x)\,u + h(t, x)],\ x(0) = q^{\circ} \qquad (3.4)$$

$$g(t, x) \equiv Q^{-1}(t)\,G(t, q^* + Qx),\quad h(t, x) \equiv Q^{-1}(t)\,L(t, q^* + Qx)$$

where functions g and h are quasiperiodic in t with frequencies of the form $\{\Omega, \Omega - \nu, \Omega + \nu\}$ and their combinations with integral multipliers whose magnitude does not exceed k. The coefficients of trigonometric polynomials are polynomials in x of power not higher than k. The respective boundary conditions for vector $x(T)$ assume in conformity with (3.3) the form $M(x(T)) = 0$, where $M(x) = S(q^* + Qx)$. Formula (3.1) for the functional J remains unchanged.

To solve the obtained problem we apply the necessary conditions of the maximum principle according to which the optimal control $u(t)$ and the phase trajectory $x(t)$ satisfy the relations

$$H = -(\varepsilon/2)\ u^2 + \varepsilon\,(p'gu + p'h) \to \max,\ |u| < \infty, \qquad (3.5)$$

$$u = g'(t, x)\,p,\ g' = (g_{ij})',\quad i = 1,\ldots, n,\ j = 1,\ldots, m$$

$$x^{\cdot} = \varepsilon\,[gg'p + h(t, x)],\ x(0) = q^{\circ},\ M(x(T)) = 0,\quad p^{\cdot} = -\varepsilon\left(p'\frac{\partial g}{\partial x}g'p - p'\frac{\partial h}{\partial x}\right),\quad p(T) = \left(\lambda'\frac{\partial M}{\partial x}\right)_T$$

where H is the Hamiltonian of the problem, p is the conjugate n-vector, and λ is the vector of Lagrange multipliers of dimension $r \leqslant n$. Let us assume that r is the maximum rank of matrix $(\partial M / \partial x)_T$, and determine from some r linear equations, for instance the first of the r, vector $\lambda : \lambda^* = p^{(r)}(T)\,[(\partial M / \partial x)_T^{(r)}]^{-1}$ which we substitute into the remaining $(n - r)$ transversality conditions for $p(T)$. We obtain $(n - r)$ relations linking $x(T)$ and $p(T)$: $p^{n-r}(T) = \lambda^*(\partial M / \partial x)_T^{(n-r)}$. If $n = r$ these relations are absent, and vector λ need not be determined then.

As the result, we have a boundary value problem of the type of (1.1) for a Hamiltonian system of standard form. It can be analyzed by the methods described in Sects.1 and 2, since it is of the form of a trigonometric polynomial with a finite set of frequencies. As already indicated, the application of the method of averaging to system (3.5) is convenient because the averaged system remains Hamiltonian for which the energy integral

$$\langle H\rangle = (\varepsilon / 2)\ p'\Gamma p + \varepsilon p'h = \text{const},\ \Gamma \equiv \langle g'g\rangle = \Gamma'$$

remains unchanged in the first approximation with respect to ε. In this formula angle brackets denote the averaging of respective expressions with respect to the explicitly appearing t. Averages exist and are smooth functions of the slow variables x and p.

Let us consider the perticular case when the averaged system (3.5) is linear with respect to x and p, which occurs when

$$g = g(t),\quad h = h^{(0)}(t) + \Lambda(t)\,x + h^{(2)}(t, x),\quad \langle h^{(2)}\rangle \equiv 0$$

The averaged equations have then constant coefficients. The sought solution of the two-point problem reduces to algebraic and finite equations. Let, for instance, it be required to transfer the phase vector of the input system (3.1) to the final state $q(T) = q_T$, where q_T is a specified vector, i.e. $S = q - q_T$. In conformity with (3.3) the boundary conditions (3.5) assume the form $x(0) = q^\circ$, $x(T) = x_T$, where $x_T = Q^{-1}(T)\,[q_T - q^*(T)]$ is a known vector. Let us also assume that the fundamental matrix $W(\tau)$ $(\tau = \varepsilon t)$ of the linear system with constant coefficients $w^\cdot = \varepsilon\Lambda_0 w$ where $\Lambda_0 = \langle \Lambda \rangle$ has been constructed. Then the averaged variables $\xi = \langle x \rangle$, $\eta = \langle p \rangle$ can be represented in the form

$$\xi(\tau) = W(\tau)\,[a + R(\tau)b + Z(\tau)\,h_0], \quad \eta(\tau) = W'(-\tau)b \tag{3.6}$$

$$R(\tau) = \int_0^\tau W(-s)\,\Gamma(s)\,W'(s)\,ds, \quad Z(\tau) = \int_0^\tau W(-s)\,ds, \quad h_0 = \langle h^{(0)} \rangle$$

Formula (3.6) implies that the n-vector of a is uniquely determined and the constant n-vector of b is also uniquely determined from the boundary conditions when matrix $R(\Theta)$ is nonsingular; we then have

$$a = q^\circ, \quad b = R^{-1}(\Theta)\,[W^{-1}(\Theta)x_T - q^\circ - Z(\Theta)\,h_0] \tag{3.7}$$

Thus condition (2.7) is satisfied when $|R(\Theta)| \neq 0$, i.e. the boundary value problem admits the solution $x(t,\varepsilon), p(t,\varepsilon)$ lying in the ε-neighborhood of the constructed unique solution (3.6), (3.7) of the averaged system for all $t \in [0, T]$. Substituting the approximate solution $\xi(\tau), \eta(\tau)$ into formula (3.5) we obtain for the control u and expression of the form of a program. Substituting 0 for τ, $\Theta - \tau$ for Θ, and q for q^0 we obtain the approximately optimal control in the form of synthesis. In conformity with (3.1), (3.5), and (3.6) the minimum of functional J is

$$J_0 = \frac{1}{2}\int_0^\Theta \eta'(\tau)\,\Gamma(\tau)\,\eta(\tau)\,d\tau$$

accurate to quantities of the order of ε.

The above results can be extended to the case of systems (3.1) with slowly varying parameters.

Note that in a number of cases (see Remark 5° in Sect.2) mechanical systems with variable parameters defined by equations

$$q^{\cdot\cdot} + G(\tau)\,q^\cdot + C(\tau)\,q = F(\tau, t, y) + \varepsilon f(\tau, t, q, q^\cdot, y, u), \quad y^\cdot = \varepsilon Y(\tau, t, q, q^\cdot, y, u), \quad g(0) = q^\circ, \; q^\cdot(0) = q^{\cdot 0}, \; y(0) = y^\circ \tag{3.8}$$

can be reduced to form (1.1). In these equations G is the matrix of gyroscopic forces, $C > 0$ is the rigidity matrix, y is the slow vector of controlled parameters of the system, and F, f, and Y are quasiperiodic functions of t of the X type in (1.1). It is assumed that for $\varepsilon = 0$, $\tau, y = \text{const}$ the vector $q = q(\tau, t, a, b, c)$ is a quasiperiodic function of t for $t \geqslant 0$ and a, b, and c are taken from some convex neighborhood of point $q^\circ, q^{\cdot\circ}, y^\circ$.

Single-frequency oscillating mechanical systems are defined in a quasilinear approximation by the particular case of Eqs. (3.8). Examples of the latter are provided by linear oscillator systems (including multidimensional plane of three-dimensional) with the equilibrium position controlled by the displacement velocity. The equations of quasilinear oscillators define two- or three-dimensional oscillations of a pendulum with variable suspension point and slowly varying length, etc. In the case of single-frequency oscillations arbitrary dependence of perturbations on arguments is admissible, since after the substitution in the equations of osculating variables of the x type, function X in (1.1) becomes 2π-periodic with respect to the variable θ (see (2.12).

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